If, $$E(\lambda h) = 1 + \lambda h + \frac{1}{2!}(\lambda h)^2 + \dots + \frac{(\lambda h)^p}{p!}$$

then (1.26) becomes

$$y_{n+1} = y(t_{n+1}) + 0(h^{p+1}) \tag{1.27}$$

The integer p is called the order of the method (1.26). The remainder term

$$\frac{(\lambda h)^{p+1}}{(p+1)!} e^{\theta\lambda h}, \quad 0 < \theta < 1$$

which is neglected, is the *relative discretization* or *local truncation error*.

The numerical methods for finding solution of the initial value problem of Equation (1.8) may broadly be classified into the following two types:

(i) *Singlestep methods* These methods enable us to find approximation to the true solution $y(t)$ at t_{n+1} if y_n, y'_n and h are known.

(ii) *Multistep methods* These methods use recurrence relations, which express the function value $y(t)$ at t_{n+1} in terms of the function values $y(t)$ and derivative values $y'(t)$ at t_{n+1} and at previous nodal points.

It is obvious from (1.27) that the numerical methods of order p will produce exact results for all differential equations whose solutions are polynomials of degree p or less. If

$$y(t) = a_0 + a_1 t$$

where a_0 and a_1 are arbitrary constants, then the singlestep method of order one will be recurrence relation between the values y_{n+1}, y_n and y'_n. We may write

$$y_{n+1} = a_0 + a_1 t_{n+1}$$
$$y_n = a_0 + a_1 t_n$$
$$y'_n = a_1$$

Eliminating a_0 and a_1, we obtain

$$y_{n+1} = y_n + h y'_n$$

Thus the singlestep numerical method of order one for Equation (1.8) will be of the form

$$y_{n+1} = y_n + h f_n, \; n = 0, 1, 2, \dots, N-1$$

where $$y'_n = f_n = f(t_n, y_n)$$

The exact values of $y(t)$ will satisfy

$$y(t_{n+1}) = y(t_n) + hf(t_n, y(t_n)) + T_n \tag{1.28}$$

where T_n is the local truncation error of the form

$$T_n = C_2 h^2 y''(\xi_2), \; t_n < \xi_2 < t_{n+1}$$

To determine C_2, we substitute $y(t) = t^2$ in (1.28), and get $C_2 = 1/2$.

Next, we construct a multistep numerical method that will produce exact results whenever $y(t)$ is a polynomial of degree three or less. Consider

$$y(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

where a_0, a_1, a_2 and a_3 are arbitrary constants.

A simple third order multistep method uses a recurrence relation between the values y_{n+1}, y_n, y'_n, y'_{n-1} and y'_{n-2}. Eliminating a_0, a_1, a_2 and a_3 from the equations

$$\begin{aligned}
y_{n+1} &= a_0 + a_1 t_{n+1} + a_2 t_{n+1}^2 + a_3 t_{n+1}^3 \\
y_n &= a_0 + a_1 t_n + a_2 t_n^2 + a_3 t_n^3 \\
y'_n &= a_1 + 2a_2 t_n + 3a_3 t_n^2 \\
y'_{n-1} &= a_1 + 2a_2 t_{n-1} + 3a_3 t_{n-1}^2 \\
y'_{n-2} &= a_1 + 2a_2 t_{n-2} + 3a_3 t_{n-2}^2
\end{aligned}$$

we obtain

$$y_{n+1} = y_n + \frac{h}{12}(23y'_n - 16y'_{n-1} + 5y'_{n-2})$$

The third order multistep method for Equation (1.8) becomes

$$y_{n+1} = y_n + \frac{h}{12}(23f_n - 16f_{n-1} + 5f_{n-2}), \quad n = 2, 3, \dots, N-1 \tag{1.29}$$

Here we need y_0, y_1 and y_2 initially to start the computation. The exact values of $y(t)$ will satisfy

$$y(t_{n+1}) = y(t_n) + \frac{h}{12}[23f(t_n, y(t_n)) - 16f(t_{n-1}, y(t_{n-1})) + 5f(t_{n-2}, y(t_{n-2}))] + T_n \tag{1.30}$$

where T_n is the local truncation error given by

$$T_n = C_4 h^4 y^{(4)}(\xi), \quad t_{n-2} < \xi < t_{n+1} \tag{1.31}$$

Putting $y(t) = t^4$ in (1.30), we get $C_4 = 3/8$.

In an analogous manner, we may obtain numerical methods based on functions other than polynomials.

Example 1.1 Find the numerical solution of the initial value problem

$$\begin{aligned}
y' &= \lambda y, \quad \lambda = \pm 1, \\
y(0) &= 1, \quad 0 \leqslant t \leqslant 2
\end{aligned}$$

Use the first order method

$$y_{n+1} = (1 + \lambda h) y_n, \quad n = 0, 1, 2, \dots, N-1$$

with $h = .1$.

We obtain

$$\begin{aligned}
y_1 &= (1 + .1\lambda) y_0 = (1 + .1\lambda) \\
y_2 &= (1 + .1\lambda) y_1 = (1 + .1\lambda)^2
\end{aligned}$$

$$y_3 = (1+.1\lambda)\, y_2 = (1+.1\lambda)^3$$
$$\vdots$$
$$y_N = (1+.1\lambda)\, y_{N-1} = (1+.1\lambda)^N$$

where $N = 20$.

The values of y_n for $\lambda = \pm 1$ are listed in Table 1.2, together with the true values obtained from $y(t_n) = e^{\lambda nh}$. These values are plotted in Figure 1.1 and compared with the true values. We observe that for $\lambda = 1$, the approxi mate solution increases as fast as the exact solution whereas for $\lambda = -1$ the approximate solution decreases at least as fast as the exact solution.

TABLE 1.2 SOLUTION OF $y' = \lambda y$, $y(0) = 1$, $0 \leq t \leq 2$ WITH $h = 0.1$

	$\lambda = 1$		$\lambda = -1$	
t	First order method	Exact solution	First order method	Exact solution
0	1	1	1	1
0.2	1.21000	1.22140	0.81	0.81873
0.4	1.46410	1.49182	0.6561	0.67032
0.6	1.77156	1.82212	0.53144	0.54881
0.8	2.14359	2.22554	0.43047	0.44933
1.0	2.59374	2.71828	0.34868	0.36788
1.2	3.13843	3.32012	0.28243	0.30119
1.4	3.79750	4.05520	0.22877	0.24660
1.6	4.59497	4.95303	0.18530	0.20190
1.8	5.55992	6.04965	0.15009	0.16530
2.0	6.72750	7.38906	0.12158	0.13534

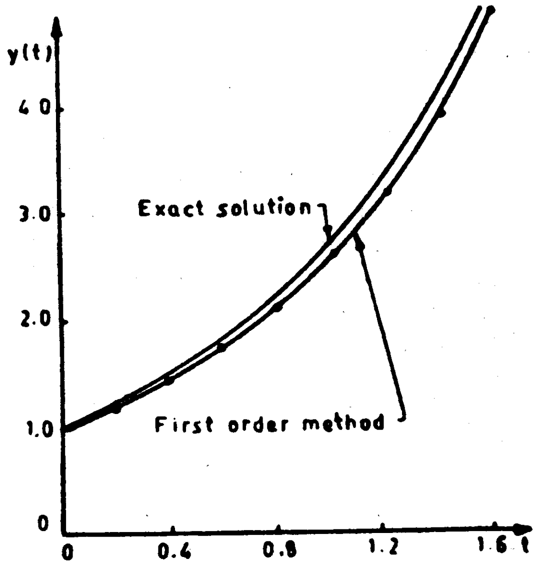

Fig. 1.1 (a) Numerical solution of $y' = y$, $y(0) = 1$

independent solutions of the homogeneous equation with $T_n = 0$ in (1.37). The homogeneous equation becomes

$$\epsilon_{n+1} = \epsilon_n + \frac{\bar{h}}{12}(23\epsilon_n - 16\epsilon_{n-1} + 5\epsilon_{n-2}) \tag{1.38}$$

where $\bar{h} = \lambda h$.

We look for the solution of this equation in the form

$$\epsilon_n = A\,\xi^n \tag{1.39}$$

where $A \neq 0$, ξ is an arbitrary number to be determined. Substituting (1.39) in (1.38) and simplifying, we obtain

$$\xi^3 - \left(1 + \frac{23}{12}\bar{h}\right)\xi^2 + \frac{4}{3}\bar{h}\xi - \frac{5}{12}\bar{h} = 0 \tag{1.40}$$

If ξ_{1h}, ξ_{2h} and ξ_{3h} are the three roots (distinct) of the characteristic equation, then the solution of the difference equation (1.38) is of the form

$$c_1\,\xi_{1h}^n + c_2\,\xi_{2h}^n + c_3\,\xi_{3h}^n$$

Suppose the characteristic equation has a double root,

$$\xi_{2h} = \xi_{3h},\ \xi_{1h} \neq \xi_{2h}$$

then the form of the above solution is modified to

$$c_1\,\xi_{1h}^n + (c_2 + c_3\,n)\xi_{2h}^n$$

If $\xi_{1h} = \xi_{2h} = \xi_{3h}$, then the solution of the difference equation is of the type

$$(c_1 + c_2\,n + c_3\,n^2)\xi_{1h}^n$$

To obtain the particular solution of the inhomogeneous equation (1.37), we assume $T_n = T$, a constant; then we find the particular solution as $T/\bar{h}$.

The general solution of (1.37) for distinct roots becomes

$$\epsilon_n = c_1\,\xi_{1h}^n + c_2\,\xi_{2h}^n + c_3\,\xi_{3h}^n + \frac{T}{\bar{h}} \tag{1.41}$$

where c_1, c_2 and c_3 are arbitrary constants to be determined from the initial errors. For stability, $|\epsilon_n| < \infty$ as $n \to \infty$ and if any $|\xi_{ih}| > 1$, the error $|\epsilon_n|$ increases unboundedly. If two or more ξ_{ih} are equal and equal to one, then also $|\epsilon_n|$ increases unboundedly.

DEFINITION 1.5 A multistep method when applied to $y' = \lambda y$, $\lambda < 0$, is said to be absolutely stable if the roots of the characteristic equation of the homogeneous difference equation for the error are either inside the unit circle or on the unit circle and simple.

The roots of Equation (1.40) are plotted in Figure 1.2. In the graph the roots are displayed in the following fashion: for real roots the absolute value of the roots is plotted, and for conjugate complex roots the modulus of the pair is plotted as a single quantity (thus conjugate pair of roots are shown as a single curve). In Figure 1.2, it can be seen that $|\xi_{2h}|$ is greater than one at $\bar{h} = -0.55$ and in Equation (1.41) the term containing $|\xi_{2h}|$ grows without bound as n gets large. Thus the third order method (1.29) is absolutely stable for $-0.55 < \lambda h < 0$.

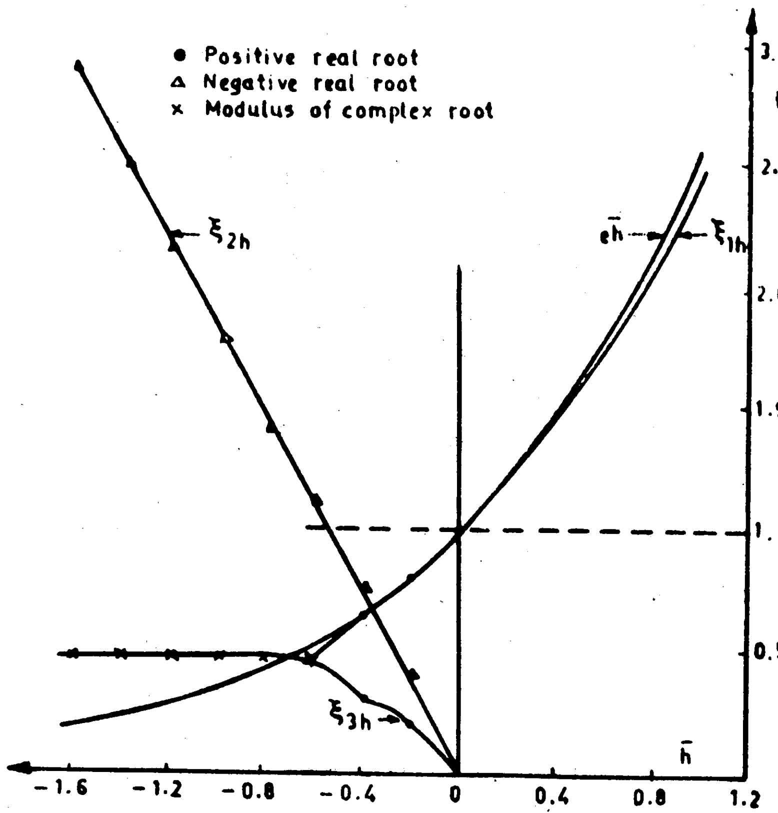

Fig. 1.2 Roots of the characteristic equation of the third order method

1.5.1 Interval of Absolute Stability

In the previous section we have determined roots of the characteristic equation (1.40) by repeatedly solving the polynomial equation for a range of values of λh. A plot of the roots against λh (see Figure 1.2) then enables us to obtain the interval of absolute stability as $(-.55, 0)$. This procedure is known as the *root locus method.*

An alternative to this procedure consists of applying to the characteristic equation (1.40) a transformation which maps the interior of the unit circle onto the left half-plane and then using the Routh-Hurwitz criterion which

DEFINITION 1.6 A numerical method of the form (1.26) is said to be *convergent* if

$$\lim_{h\to 0} y_n = y(t_n) \text{ for all } t_n \in [t_0, b]$$

$$t_n = t_0 + nh$$

The true value $y(t_n)$ will satisfy

$$y(t_{n+1}) = E(\lambda h) y(t_n) + T_n \tag{1.45}$$

where T_n denotes the truncation error.

The approximate solution will satisfy

$$y_{n+1} = E(\lambda h) y_n - R_n \tag{1.46}$$

where R_n denotes the round-off error.

Subtracting (1.45) from (1.46) and by substituting $\epsilon_n = y_n - y(t_n)$, we get

$$\epsilon_{n+1} = E(\lambda h)\epsilon_n - R_n - T_n \tag{1.47}$$

Let us denote $\max\limits_{(t_0, t_N)} |R_n| = R$ and $\max\limits_{(t_0, t_N)} |T_n| = T$ and assume these as constants. Then (1.47) becomes

$$|\epsilon_{n+1}| \leqslant E(\lambda h) |\epsilon_n| + R + T, \; n = 0, 1, 2, \ldots \tag{1.48}$$

By induction, we can write (1.48) for $E(\lambda h) \neq 1$ as

$$|\epsilon_n| \leqslant E^n(\lambda h) |\epsilon_0| + \frac{E^n(\lambda h) - 1}{E(\lambda h) - 1}(R+T) \tag{1.49}$$

Let $E(\lambda h)$ be the pth order approximation, then

$$e^{\lambda h} = E(\lambda h) + \frac{(\lambda h)^{p+1}}{(p+1)!} M_{p+1}$$

where M_{p+1} is a constant. Thus (1.49) becomes

$$|\epsilon_n| \leqslant |\epsilon_0| \, e^{\lambda(t_n - t_0)} + \frac{e^{\lambda(t_n - t_0)} - 1}{\lambda\left(1 + \frac{\lambda h}{2!} + \frac{\lambda^2 h^2}{3!} + \ldots + \frac{\lambda^{p-1} h^{p-1}}{p!}\right)} \times \left(\frac{R}{h} + \frac{\lambda^{p+1} h^p}{(p+1)!} M_{p+1}\right) \tag{1.50}$$

It is obvious that in the absence of the initial and round-off errors, $|\epsilon_n| \to 0$ as $h \to 0$ like Ch^p where C is a constant, independent of h.

For $|\epsilon_0| = 0$ and $p = 1$, (1.50) becomes

$$|\epsilon_n| \leqslant \frac{e^{\lambda(t_n - t_0)} - 1}{\lambda}\left(\frac{R}{h} + \frac{\lambda^2 h}{2} M_2\right) \tag{1.51}$$

The dependence of $|\epsilon_n|$ on h is shown in Figure 1.4. Clearly, as $h \to 0$, the truncation error tends to zero whereas the round-off error becomes infinite. On the other hand as $h \to \infty$, the round-off error tends to zero but the

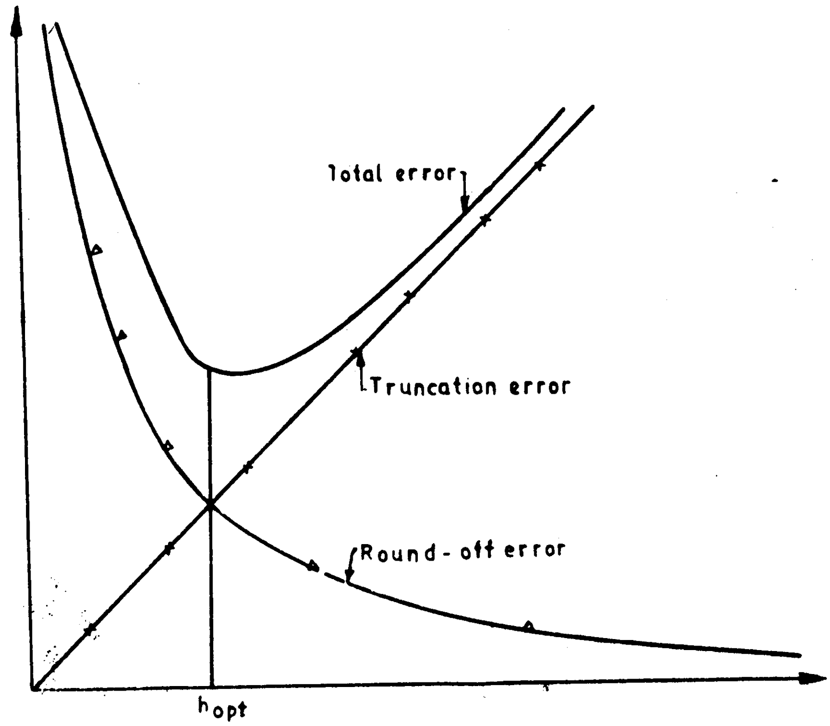

Fig. 1.4 Truncation and round-off errors as function of h

truncation error becomes infinitely large. The choice of h for which the bound (1.51) is minimum is obtained when

$$h = \sqrt{\frac{2R}{\lambda^2 M_2}}$$

We also observe from (1.51) that the first order method converges as $h \to 0$ if the round-off error is of order h^2.

To discuss the convergence of the multistep method (1.29), we determine the constants c_1, c_2, c_3 in (1.41). Let us denote

$$E_j = \epsilon_j - \frac{T}{h}, \quad j = 0, 1, 2$$

The constants c_1, c_2, c_3 can be found by solving the linear system

$$\begin{aligned} E_0 &= c_1 + c_2 + c_3, \\ E_1 &= c_1 \xi_{1h} + c_2 \xi_{2h} + c_3 \xi_{3h}, \\ E_2 &= c_1 \xi_{1h}^2 + c_2 \xi_{2h}^2 + c_3 \xi_{3h}^2 \end{aligned}$$

These results show that for this method, the magnitude of the truncation error coefficient C_2 decreases towards zero (from the right) and then increases negatively as the magnitude of $h\lambda_{max}$ decreases to zero. The values of C_2 and $h\lambda_{max}$ are shown in Figure 1.5.

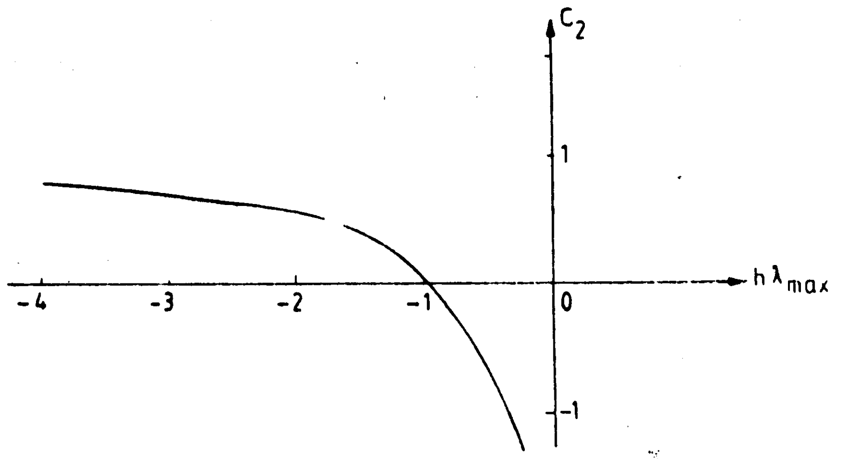

Fig. 1.5 Truncation error coefficient C_2 as function of $h\lambda_{max}$

Bibliographical Note

There are many numerical analysis books having chapters concerning numerical solution of ordinary differential equations, e.g. 61, 103, 114, 121 and 237. Some useful books which deal with the numerical methods for ordinary differential equations in detail are 33, 93, 113, 161 and 163. The difference methods for ordinary and partial differential equations are given in 46, 88 and 147.

Problems

1. Prove that each of the following ordinary differential equation has a unique solution on the interval indicated:
   (i) $y' = t^2 \exp(-y^2)$, $y(0) = 1$ on $[0, 10]$
   (ii) $y' = ty + t^2$, $y(0) = 0$ on $[0, \frac{1}{2}]$
2. Verify that the function

$$y(t) = y_0 \exp\left(\int_{t_0}^{t} p(\tau)\, d\tau\right) + \exp\left(\int_{t_0}^{t} p(\tau)\, d\tau\right) \int_{t_0}^{t} \left[g(\tau) \exp\left(-\int_{t_0}^{\tau} p(s)\, ds\right)\right] d\tau$$

satisfies the initial value problem

$$\frac{dy}{dt} = p(t)y+g(t),\ y(t_0) = y_0$$

3. (i) Solve the initial value problem
$y'-ky = A \sin wt,\ y(0) = y_0$
where k, A, w and y_0 are given constants.
(ii) Assuming that $k < 0$ show that the steady-state response is equal to $A_0 \sin (wt-\phi)$ where $A_0 = \frac{A}{\sqrt{k^2+w^2}}$ and $\tan \phi = \frac{w}{k} \cdot \frac{A_0}{A}$ is called the *amplification factor*, and ϕ is called the *phase angle.*

4. Consider a system of first order ordinary differential equations

$$\frac{dv_i}{dt} = f_i(t, v_1, v_2, \ldots, v_m)$$

$$v_i(t_0) = v_{i0} \qquad i = 1(1)m$$

Assume that each of the functions $f_i(t, v_1, v_2, \ldots, v_m)$ is continuous and bounded and satisfies a Lipschitz condition in

$v_1, v_2, \ldots, v_m$ for $t \in [t_0, b]$ and $-\infty < v_1, v_2, \ldots, v_m < \infty$

Then the system of the first order equations has a unique solution on $[t_0, b]$.
Investigate the existense and uniqueness of a solution to the following system on [0, 2]:

$$v_1' = 3t+4tv_1-v_2+v_3, \qquad v_1(0) = 1$$

$$v_2' = t \exp(-v_2^2), \qquad v_2(0) = -1$$

$$v_3' = t^2+\frac{1}{(t-4)}(v_1+v_2+v_3), \qquad v_3(0) = 1$$

5. Consider the following system of two simultaneous second order equations

$$u'' = g_1(t, u, v, u', v')$$

$$v'' = g_2(t, u, v, u', v')$$

$$u(t_0) = u_0,\ u'(t_0) = u_0'$$

$$v(t_0) = v_0,\ v'(t_0) = v_0'$$

(i) Convert the above system into a system of first order equations.
(ii) State the number of first order equations in the system.

6. Find y_n from the difference equation
$\Delta^2 y_{n+1}+\frac{1}{4}\Delta^2 y_n = 0,\ n = 0, 1, 2, \ldots$
when $y_0 = 0, y_1 = \frac{1}{2}, y_2 = \frac{1}{4}$. (BIT7 (1967), 81)

Estimate the parameter w by requiring that the method be also exact for the following functions:

(i) $y(t) = \{1, t, t^2\}$

(ii) $y(t) = \{1, t, \exp(-wt)\}$

22. Consider the recursion formula:

$$y_{n+1} = y_{n-1} + 2hy_n$$
$$y_0 = 1$$
$$y_1 = 1 + h + h^2\left(\frac{1}{2} + \frac{h}{6} + \frac{h^2}{24}\right)$$

Show that

$y_n - e^{nh} = 0(h^2)$ as $h \to 0$, nh = constant. (BIT14 (1974), 482)

23. Show that the general solution of the initial value problem

$y'' + y = r(t)$, $y(0) = y_0$, $y'(0) = y'_0$ can be written as

$$y(t) = y_0 \cos t + y'_0 \sin t + \int_0^t r(\tau) \sin(t-\tau)\, d\tau$$

24. Show that the numerical solution of the initial value problem

$$y'' + y = 0,\ y(0) = 1,\ y'(0) = 0$$

may be obtained by satisfying the formula

$$y_{n+1}^2 + y_{n+1}'^2 = y_n^2 + y_n'^2$$

25. Consider the following system of difference equations:

$$\begin{bmatrix} u_n \\ v_n \end{bmatrix} = \begin{bmatrix} -1+\cos(h) & \sin(h) \\ -\sin(h) & -1+\cos(h) \end{bmatrix} \begin{bmatrix} u_{n-1} \\ v_{n-1} \end{bmatrix}$$

How small should h be chosen so that $u_n \to 0$ and $v_n \to 0$ when $n \to \infty$.

(BIT20 (1980), 389)

2

Singlestep Methods

2.1 INTRODUCTION

A singlestep method for the solution of the differential equation

$$\frac{dy}{dt} = f(t, y),\ y(t_0) = y_0,\ t \in [t_0, b] \tag{2.1}$$

is one in which the solution of the differential equation is approximated by calculating the solution of a related first order difference equation. Thus, a general singlestep method can be written in the form

$$y_{n+1} = y_n + h\phi(t_n, y_n, h),\ n = 0, 1, 2, \cdots, N-1 \tag{2.2}$$

where $\phi(t, y, h)$ is a function of the arguments t, y, h and, in addition, depends on the right-hand side of (2.1). The function $\phi(t, y, h)$ is called the *increment function*. If y_{n+1} can be obtained simply by evaluating the right-hand side of (2.2), then the singlestep method is called explicit otherwise it is called implicit. The true value $y(t_n)$ will satisfy

$$y(t_{n+1}) = y(t_n) + h\phi(t_n, y(t_n), h) + T_n,\ n = 0, 1, 2, \ldots, N-1 \tag{2.3}$$

where T_n is the truncation error.

The largest integer p such that $| h^{-1} T_n | = 0(h^p)$ is called the *order* of the singlestep method.

Before stating the main result about convergence, we introduce a few definitions.

DEFINITION 2.1 The singlestep method (2.2) is said to be *regular* if the function $\phi(t, y, h)$ is defined and continuous in the domain $t_0 \leqslant t \leqslant b$, $-\infty < y < \infty$, $0 \leqslant h \leqslant h_0$ (h_0 is a positive constant) and if there exists a constant L such that

$$| \phi(t, y, h) - \phi(t, z, h) | \leqslant L \, | y - z | \tag{2.4}$$

for every $t \in [t_0, b]$, $y, z \in (-\infty, \infty)$, $h \in (0, h_0)$.

DEFINITION 2.2 A singlestep method of the form (2.2) is said to be *consistent* if

$$\phi(t, y, 0) = f(t, y)$$

We must also ensure that the formula (2.2) be insensitive to small change in the local errors. This will be guaranteed by the stability condition. The main result of convergence is

THEOREM 2.1 *A necessary and sufficient condition for convergence of a regular singlestep method of order $p \geqslant 1$ is consistency.*

This result ensures that the approximate solution converges to the exact solution like $C\,h^p$.

For the application of the formula (2.2) to (2.1), we need a specific form of the increment function $\phi\,(t,\,y,\,h)$.

2.2 TAYLOR SERIES METHOD

Let us assume that the differential equation (2.1) has a unique solution $y(t)$ on $[t_0, b]$ and that $y(t) \in C^{(p+1)}[t_0, b]$ for $p \geqslant 1$. The solution $y(t)$ can be expanded in a Taylor series about any point t_n

$$y\,(t) = y\,(t_n)+(t-t_n)y'\,(t_n)+\frac{1}{2\,!}(t-t_n)^2\,y''\,(t_n)+\cdots$$
$$+\frac{1}{p\,!}\,(t-t_n)^p\,y^{(p)}\,(t_n)+\frac{(t-t_n)^{p+1}}{(p+1)!}\,y^{(p+1)}\,(\xi_n) \tag{2.5}$$

This expansion holds good for $t \in [t_0, b]$; $t_n < \xi < t$. Substituting $t = t_{n+1}$ in (2.5), we get

$$y\,(t_{n+1}) = y\,(t_n)+hy'\,(t_n)+\frac{h^2}{2\,!}y''\,(t_n)+\;\cdots+\frac{h^p}{p\,!}y^{(p)}\,(t_n)+\frac{h^{p+1}}{(p+1)\,!}y^{(p+1)}\,(\xi_n)$$

We define

$$h\,\phi\,(t_n,\,y\,(t_n),\,h)=h\,y'(t_n)+\frac{h^2}{2\,!}y''\,(t_n)+\cdots+\,\frac{h^p}{p\,!}\,y^{(p)}\,(t_n)$$

and $h\,\phi\,(t_n,\,y_n,\,h)$ is to be obtained from $h\,\phi\,(t_n,\,y\,(t_n),\,h)$ by using an approximate value y_n in place of the exact value $y(t_n)$. We compute

$$y_{n+1} = y_n+h\phi(t_n,\,y_n,\,h),\; n = 0,\,1,\,2,\,\ldots,\,N-1 \tag{2.6}$$

to approximate $y\,(t_{n+1})$. This is called *Taylor's series method* of order p. Substituting $p = 1$ in (2.6), we get

$$y_{n+1} = y_n+hf\,(t_n,\,y_n),\; n = 0,\,1,\,2,\,\ldots,\,N-1 \tag{2.7}$$

which is known as *Euler's method.* To apply (2.6), it is necessary to know $y\,(t_n)$, $y'\,(t_n)$, $\ldots$, $y^{(p)}\,(t_n)$. If t_n and $y\,(t_n)$ were known, the derivatives can be calculated as follows:

First the known values t_n and $y(t_n)$ are substituted into the differential equation to give

$$y'\,(t_n) = f\,(t_n,\,y\,(t_n))$$

Next, the differential equation (2.1) can be differentiated to obtain formulas for the higher derivatives of $y\,(t)$.

Thus
$$y' = f(t, y)$$
$$y'' = f_t + f f_y$$
$$y''' = f_{tt} + 2f f_{ty} + f^2 f_{yy} + f_y(f_t + f f_y)$$
$$\vdots$$

where $f_t, f_y, \ldots$ represent the derivatives of f with respect to t and y.

The values $y''(t), y'''(t), \ldots$ can be computed by substituting $t = t_n$. Therefore, if t_n and $y(t_n)$ were known exactly, then (2.6) could be used to compute $y(t_{n+1})$ with an error

$$\frac{h^{p+1}}{(p+1)!}\, y^{(p+1)}(\xi_n),\ t_n < \xi_n < t_{n+1}$$

The number of terms to be included in (2.6) is fixed by the permissible error. If this error is ϵ and the series is truncated at the term $y^{(p)}(t_n)$, then

$$h^{p+1}\,|\,y^{(p+1)}(\xi_n)\,| < (p+1)!\,\epsilon$$

or

$$h^{p+1}\,|\,f^{(p)}(\xi_n)\,| < (p+1)!\,\epsilon \tag{2.8}$$

For given h, (2.8) will determine p, and if p is specified, it will give an upper bound on h. The value $|\,f^{(p)}(\xi_n)\,|$ in (2.8) may be replaced by max. $|\,f^{(p)}(t_n)\,|$ in $[t_0, b]$ for computational purposes.

Example 2.1 Solve the differential equation

$$y' = t + y,\ y(0) = 1,\ t \in [0, 1]$$

by Taylor's series method, and determine the number of terms to be included in Taylor's series to obtain an accuracy of 10^{-10}.

The high order derivatives of $y(t)$ can be calculated by successively differentiating the differential equation

$$y' = t + y$$

We get
$$y'' = 1 + y'$$

and
$$y^{(r+1)} = y^{(r)},\ r = 2, 3, \ldots$$

Also we have

$$y'(0) = 1$$
$$y''(0) = 2$$
$$y^{(r)}(0) = 2,\ r = 3, 4 \ldots$$

Therefore, we obtain

$$y(t) = 1 + t + t^2 + \frac{2t^3}{3!} + \ldots + \frac{2t^p}{p!}$$

To get results accurate up to 10^{-10} for $t \leqslant 1$, we obtain from (2.8)

$$\frac{2e}{(p+1)!} < 5 \times 10^{-11}$$

which gives $p \cong 15$. Hence it follows that about 15 terms are required to achieve the accuracy in the range $t \leqslant 1$.

and thus we have the approximation

$$y_{n+1} = y_n+\frac{h}{2}[f(t_n, y_n)+f(t_{n+1}, y_n+hf(t_n, y_n))] \quad (2.15)$$

Either (2.14) or (2.15) can be regarded as

$$y_{n+1} = y_n+h \text{ (average slope)} \quad (2.16)$$

This is the underlying idea of the Runge-Kutta approach. In general, we find the slope at t_n and at several other points: average these slopes, multiply by h, and add the result to y_n. Thus the Runge-Kutta method with v slopes can be written as

$$K_i = hf(t_n+c_ih, y_n+\sum_{j=1}^{i-1} a_{ij}K_j), \; c_1 = 0, \; i = 1, 2,..., v \quad (2.17)$$

or

$$\begin{aligned} K_1 &= hf(t_n, y_n) \\ K_2 &= hf(t_n+c_2h, y_n+a_{21} K_1) \\ K_3 &= hf(t_n+c_3h, y_n+a_{31} K_1+a_{32} K_2) \\ K_4 &= hf(t_n+c_4h, y_n+a_{41} K_1+a_{42} K_2+a_{43} K_3) \\ &\vdots \end{aligned}$$

and

$$y_{n+1} = y_n+\sum_{i=1}^{v} w_i K_i$$

where the parameters $c_2, c_3, \ldots, c_v, a_{2j} \ldots, a_{v(v-1)}$ and w_i are arbitrary.

From (2.16), we may interpret the increment function as the linear combination of the slopes at t_n and at several other points between t_n and t_{n+1}. To obtain specific values for the parameters, we expand y_{n+1} in powers of h such that it agrees with the Taylor series expansion of the solution of the differential equation to a specified number of terms.

Let us study this approach with just two slopes.

2.3.1 Second Order Methods

Define

$$K_1 = hf(t_n, y_n)$$

$$K_2 = hf(t_n+c_2h, y_n+a_{21} K_1)$$

and

$$y_{n+1} = y_n+w_1 K_1+w_2 K_2 \quad (2.18)$$

where the parameters c_2, a_{21}, w_1 and w_2 are chosen to make y_{n+1} closer to $y(t_{n+1})$.

Now Taylor's series gives

$$y(t_{n+1}) = y(t_n)+hy'(t_n)+\frac{h^2}{2!}y''(t_n) + \frac{h^3}{3!}y'''(t_n)+\ldots \quad (2.19)$$

where

$$\begin{aligned} y' &= f(t, y) \\ y'' &= f_t+ff_y \\ y''' &= f_{tt}+2ff_{ty}+f_{yy}f^2+f_y(f_t+ff_y) \end{aligned}$$

The values of $y'(t_n)$, $y''(t_n)$, ... are obtained by substituting $t = t_n$. We expand K_1 and K_2 about the point (t_n, y_n).

$$
\begin{aligned}
K_1 &= hf_n \\
K_2 &= hf(t_n+c_2h,\ y_n+a_{21}hf_n) \\
&= h[f(t_n, y_n)+(c_2hf_t+a_{21}hf_nf_y)+ \\
&\quad \frac{1}{2!}(c_2^2h^2f_{tt}+2c_2a_{21}h^2f_nf_{ty}+a_{21}^2h^2f_n^2f_{yy})+\dots] \\
&= hf_n+h^2(c_2f_t+a_{21}f_nf_y)+ \\
&\quad \frac{1}{2}h^3(c_2^2f_{tt}+2c_2a_{21}f_nf_{ty}+a_{21}^2f_n^2f_{yy})+\dots
\end{aligned}
$$

Substituting the values of K_1 and K_2 in (2.18), we get

$$
\begin{aligned}
y_{n+1} &= y_n+(w_1+w_2)hf_n+h^2(w_2c_2f_t+w_2a_{21}f_nf_y)+ \\
&\quad \frac{1}{2}h^3w_2(c_2^2f_{tt}+2c_2a_{21}f_nf_{ty}+a_{21}^2f_n^2f_{yy})+\dots
\end{aligned}
\tag{2.20}
$$

Comparing (2.19) with (2.20) and matching coefficients of powers of h, we obtain three equations for the parameters

$$
\begin{aligned}
w_1+w_2 &= 1 \\
c_2w_2 &= 1/2 \\
a_{21}w_2 &= 1/2
\end{aligned}
$$

From these equations, we see that if c_2 is chosen arbitrarily (nonzero), then

$$a_{21} = c_2,\ w_2 = \frac{1}{2c_2},\ w_1 = 1-\frac{1}{2c_2} \tag{2.21}$$

Using (2.21) in (2.20), we get

$$y_{n+1} = y_n+hf_n+\frac{h^2}{2}(f_t+f_nf_y)+\frac{c_2h^3}{4}(f_{tt}+2f_nf_{ty}+f_n^2f_{yy})+\dots \tag{2.22}$$

Subtracting (2.22) from (2.19), we obtain the local truncation error

$$
\begin{aligned}
T_n &= y(t_{n+1})-y_{n+1} \\
&= h^3\left[\left(\frac{1}{6}-\frac{c_2}{4}\right)\left(f_{tt}+2f_nf_{ty}+f_n^2f_{yy}\right)+\frac{1}{6}f_y\left(f_t+f_nf_y\right)\right]+\dots \\
&= \frac{h^3}{12}[(2-3c_2)y'''+3c_2f_yy_n'']+\dots
\end{aligned}
$$

We observe that no choice of the parameter c_2 will make the leading term of T_n vanish for all $f(t, y)$. The local truncation error depends not only on derivatives of the solution $y(t)$ but also on the function f. This is typical of all the Runge-Kutta methods; in most other methods the truncation error depends only on certain derivatives of $y(t)$. Generally, c_2 would be chosen between 0 and 1. From the definition of the Runge-Kutta equations, we see

$$\begin{array}{c|ccc} \frac{2}{3} & \frac{2}{3} & & \\ \frac{2}{3} & 0 & \frac{2}{3} & \\ \hline & \frac{2}{8} & \frac{3}{8} & \frac{3}{8} \end{array}$$

Nystrom

$$\begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & & \\ \frac{3}{4} & 0 & \frac{3}{4} & \\ \hline & \frac{2}{9} & \frac{3}{9} & \frac{4}{9} \end{array}$$

Nearly Optimal

$$\begin{array}{c|ccc} \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{array}$$

Classical

$$\begin{array}{c|ccc} \frac{1}{3} & \frac{1}{3} & & \\ \frac{2}{3} & 0 & \frac{2}{3} & \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array}$$

Heun

2.3.3 Fourth order methods

The details of the derivation of the fourth order method will be omitted, since they follow the same pattern as above.

In the above notations we can write the fourth order formulas as:

$$\begin{array}{c|cccc} \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array}$$

Classical

$$\begin{array}{c|cccc} \frac{1}{3} & \frac{1}{3} & & & \\ \frac{2}{3} & -\frac{1}{3} & 1 & & \\ 1 & 1 & -1 & 1 & \\ \hline & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$$

Kutta

$$\begin{array}{c|cccc} \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & (\sqrt{2}-1)/2 & (2-\sqrt{2})/2 & & \\ 1 & 0 & -\sqrt{2}/2 & 1+\sqrt{2}/2 & \\ \hline & \frac{1}{6} & (2-\sqrt{2})/6 & (2+\sqrt{2})/6 & \frac{1}{6} \end{array}$$

Gill

$\frac{1}{3}$	$\frac{1}{3}$				
$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$			
$\frac{1}{2}$	$\frac{1}{8}$	0	$\frac{3}{8}$		
1	$\frac{1}{2}$	0	$-\frac{3}{2}$	2	
	$\frac{1}{6}$	0	0	$\frac{2}{3}$	$\frac{1}{6}$

Merson

Example 2.3 Solve the initial value problem

$$y' = t+y,\ y(0) = 1,\ t \in [0, 1]$$

by classical fourth order Runge-Kutta method with $h = .1$.

For $n = 0$ $\quad t_0 = 0, y_0 = 1$

$$K_1 = hf(t_0, y_0) = (.1)\,(0+1) = .1$$

$$K_2 = hf\left(t_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right)$$

$$= (.1)\left[0+\frac{.1}{2}+\left(1+\frac{.1}{2}\right)\right] = .11$$

$$K_3 = hf\left(t_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right)$$

$$= (.1)\left[0+\frac{.1}{2}+\left(1+\frac{.11}{2}\right)\right] = .1105$$

$$K_4 = hf(t_0+h, y_0+K_3)$$

$$= (.1)\,[(0+.1)+(1+.1105)] = .121$$

$$y_1 = 1+\frac{1}{6}[.1+.22+.2210+.12105]$$

$$= 1.11034167$$

For $n = 1$

$$t_1 = .1, y_1 = 1.11034167$$

$$K_1 = hf(t_1, y_1)$$

$$= (.1)\,[.1+1.11034167] = .121034167$$

$$K_2 = hf\left(t_1+\frac{h}{2}, y_1+\frac{K_1}{2}\right)$$

$$= (.1)\left[\left(.1+\frac{.1}{2}\right)+\left(1.11034167+\frac{1}{2}(.121034167)\right)\right]$$

$$= .132085875$$

Sixth order methods

$\frac{1}{3}$	$\frac{1}{3}$						
$\frac{2}{3}$	0	$\frac{2}{3}$					
$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$	$-\frac{1}{12}$				
$\frac{1}{2}$	$-\frac{1}{16}$	$\frac{9}{8}$	$-\frac{3}{16}$	$-\frac{3}{8}$			
$\frac{1}{2}$	0	$\frac{9}{8}$	$-\frac{3}{8}$	$-\frac{3}{4}$	$\frac{1}{2}$		
1	$\frac{9}{44}$	$-\frac{9}{11}$	$\frac{63}{44}$	$\frac{18}{11}$	0	$-\frac{16}{11}$	
	$\frac{11}{120}$	0	$\frac{27}{40}$	$\frac{27}{40}$	$-\frac{4}{15}$	$-\frac{4}{15}$	$\frac{11}{120}$

Butcher

$\frac{1}{9}$	$\frac{1}{9}$							
$\frac{1}{6}$	$\frac{1}{24}$	$\frac{3}{24}$						
$\frac{1}{3}$	$\frac{1}{6}$	$-\frac{3}{6}$	$\frac{4}{6}$					
$\frac{1}{2}$	$-\frac{5}{8}$	$\frac{27}{8}$	$-\frac{24}{8}$	$\frac{6}{8}$				
$\frac{2}{3}$	$\frac{221}{9}$	$-\frac{981}{9}$	$\frac{867}{9}$	$-\frac{102}{9}$	$\frac{1}{9}$			
$\frac{5}{6}$	$-\frac{183}{48}$	$\frac{678}{48}$	$-\frac{472}{48}$	$-\frac{66}{48}$	$\frac{80}{48}$	$\frac{3}{48}$		
1	$\frac{716}{82}$	$-\frac{2079}{82}$	$\frac{1002}{82}$	$\frac{834}{82}$	$-\frac{454}{82}$	$-\frac{9}{82}$	$\frac{72}{82}$	
	$\frac{41}{840}$	0	$\frac{216}{840}$	$\frac{27}{840}$	$\frac{272}{840}$	$\frac{27}{840}$	$\frac{216}{840}$	$\frac{41}{840}$

Huta